Minor-I Exam for EEL306 (II-Sem 2013-14)

Time: 1 Hour **Max. Marks: 30**

Instructor: Dr. Saif Khan Mohammed, saifkm@ee.iitd.ac.in, 011-26591067

Do not turn over to the next page till you are instructed to do so.

Important Instructions:

1) Write your response in the space provided after each question on this question paper.
2) Show all steps leading to the final answer.
3) There will be partial grading for intermediate steps leading to the final answer.
4) You need not prove/derive any result which has been derived in class.
5) Write legibly and clearly state any assumptions made.
6) Extra sheets for rough work must not be submitted for evaluation.
7) You are allowed to use a sheet of paper containing important formulas.
8) Calculators with clear memory are allowed.
9) Switch off your mobile phone and place it in your bag.
10) Keep your ID cards on the desk for the invigilator to examine.

Student Name: VAIBHAV GARG **Student Entry No:** 2012EE50563

Student Signature: Vaibhav

Marks Obtained: $\frac{19.5}{30}$

Examiner Signature: 8/2/14

1) (7 Marks) Prove the Parseval's Theorem which states that for any two complex-valued signals $x(t)$ and $y(t)$

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt \;=\; \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df$$

where $X(f)$ and $Y(f)$ are the Fourier transforms of $x(t)$ and $y(t)$ respectively, and * denotes complex conjugation. (**Hint**: Consider a LTI system whose impulse response is $y^*(-t)$ and whose input is $x(t)$. Derive expression for the output $z(t)$ at time $t = 0$ in two different ways, one using the convolution integral and another by taking the inverse Fourier transform of $Z(f)$)

$h(t) = y^*(-t)$

Answer: $x(t) \rightarrow \boxed{y^*(-t)} \rightarrow z(t)$

$$z(t) = \int_{-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} y^*(-\tau)\cdot x(t-\tau)\, d\tau$$

$$z(0) = \int_{-\infty}^{\infty} y^*(-\tau)\cdot x(-\tau)\, d\tau \Rightarrow z(0) = \int_{-\infty}^{\infty} x(\tau)\cdot y^*(\tau)\, d\tau \quad \text{—(1)}$$

Similarly, we know

$$Z(f) = X(f)\cdot H(f)$$

If $y(t) \xrightarrow{\mathcal{F}} Y(f)$

$y(-t) \leftrightarrow Y(-f)$

$y^*(+t) \leftrightarrow Y^*(-f) \Rightarrow y^*(-t) \leftrightarrow Y^*(f)$

good!!

$$Z(f) = X(f).Y^*(f).$$

taking the inverse fourier transform of $Z(f)$

$$z(t) = \int_{-\infty}^{\infty} Z(f)\cdot e^{+j2\pi ft}\, df$$

$$= \int_{-\infty}^{\infty} X(f)\cdot Y^*(f)\, e^{j2\pi ft}\, df$$

$$z(0) = \int_{-\infty}^{\infty} X(f)\cdot Y^*(f)\, df \quad \text{—(2)}$$

From ① & ② $\boxed{\int_{-\infty}^{\infty} x(t).y^*(t)\, dt = \int_{-\infty}^{\infty} X(f)\cdot Y^*(f)\, df}$

2) (6 Marks) Let $x(t)$ be a real-valued band-limited passband signal whose complex envelope is denoted by $\tilde{x}(t)$. Find the value of $\frac{\int_{-\infty}^{\infty} x^2(t)\,dt}{\int_{-\infty}^{\infty} |\tilde{x}(t)|^2\,dt}$. You can assume that $X(f) = 0,\ |f \pm f_c| > W$ and that $f_c > W$. **Hint:** Use the relation $x(t) = \text{Real}\left(\tilde{x}(t)\, e^{j2\pi f_c t}\right)$

Answer:

$$x(t) = \frac{\tilde{x}(t) \cdot e^{j2\pi f_c t} + \tilde{x}^*(t) \cdot e^{-j2\pi f_c t}}{2}$$

3.5

$$X(f) = \frac{\tilde{X}(f-f_c) + \tilde{X}^*(-f-f_c)}{2}$$

by observation (also proved in class)

and $X_+(f) = X_-^*(-f)$

$$X_+(f) = X(f) = \frac{\tilde{X}(f-f_c)}{2} \quad \text{for } |f-f_c| < W$$

$$X_-(f) \quad X(f) = \frac{\tilde{X}^*(-f-f_c)}{2} \quad \text{for } |f+f_c| < W$$

$$\frac{\int_{-\infty}^{\infty} x^2(t)\,dt}{\int_{-\infty}^{\infty} |\tilde{x}(t)|^2\,dt} = \frac{\int_{-\infty}^{\infty} |X(f)|^2\,df}{\int_{-\infty}^{\infty} |\tilde{X}(f)|\,df} = \frac{\cancel{2} \times 1/4 \int_{-\infty}^{\infty} |\tilde{X}(f)|^2\,df}{\cancel{2} \times \int_{-W}^{W} |\tilde{X}(f)|^2\,df} = \boxed{1/4}$$

Ans

Used concept:

$$\int |X(f)|^2 df = \int_{-f_c-W}^{-f_c+W} |X_-(f)|^2\,df + \int_{f_c-W}^{f_c+W} |X_+(f)|^2\,df$$

Why this factor of 2?

3) (7 Marks) Find the complex envelope of the passband signal

$$x(t) = A\,(1 + k\,m(t))\,\cos(2\pi f_c t + \phi)$$

where $A > 0$, $k \in \mathbb{R}$, $\phi \in (-\pi, \pi]$ are constants. $m(t)$ is a real-valued baseband signal band-limited to $[-W, W]$. Assume $f_c > W$.

Answer:

$$x(t) = A(1 + km(t))\cos(2\pi f_c t + \phi)$$

$$= A(1 + km(t)\ \mathrm{Re}\{e^{j(2\pi f_c t + \phi)}\}$$

$$= \cancel{A(}\ \mathrm{Re}\{A(1 + km(t))\,e^{j2\pi f_c t} \cdot e^{j\phi}\}$$

$$= \mathrm{Re}\{\underbrace{A(1 + km(t)) \cdot e^{j\phi}}_{\tilde{x}(t)} \cdot e^{j2\pi f_c t}\}$$

$$\tilde{x}(t) = A(1 + km(t))\,e^{j\phi}$$

7

4) (7 Marks) Consider a channel whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = x(t) - \frac{1}{2}x(t - t_0)$$

where t_0 is a constant. Assuming the real-valued input $x(t)$ to be a band-limited passband signal (i.e., $X(f) = 0$, $|f \pm f_c| > W$), find the expression for a base band signal $h_b(t)$ (band-limited to $[-W, W]$) such that

$$\tilde{y}(t) = \int_{-\infty}^{\infty} h_b(\tau)\, \tilde{x}(t - \tau)\, d\tau.$$

Here, $\tilde{y}(t)$ and $\tilde{x}(t)$ are the complex envelopes of $y(t)$ and $x(t)$ respectively.

Answer: We know $\tilde{y}(t) = \frac{1}{2}\int_{-\infty}^{\infty} \tilde{h}(\tau)\cdot\tilde{x}(t-\tau)\, d\tau$

$\Rightarrow h_b(\tau) = \frac{1}{2}\tilde{h}(\tau)$ where $\tilde{h}(\tau) =$ complex env. of $h(t)$

~~where~~ System is L.T.I

$y(t) = x(t) - \frac{1}{2}x(t-t_0)$

2

$\cancel{\xrightarrow{\mathcal{F}} Y(f) = X(f) - \frac{1}{2}X(f)\cdot e^{-j2\pi f t_0}}$

$\cancel{Y(f) = X(f)\left[1 - \frac{e^{-j2\pi f t_0}}{2}\right]}$

$\cancel{\Rightarrow H(f) = \left[1 - \frac{e^{-j2\pi f t_0}}{2}\right]} \quad \cancel{\Rightarrow h(t) = \delta(t) - \frac{\delta(t-t_0)}{2}}$

$y(t) = \mathrm{Re}\{\tilde{y}(t)\cdot e^{j2\pi f_c t}\}$ $\quad x(t) = \mathrm{Re}\{\tilde{x}(t)\cdot e^{j2\pi f_c t}\}$

$x(t-t_0) = \mathrm{Re}\{\tilde{x}(t-t_0)\, e^{j2\pi f_c(t-t_0)}\}$

$x(t) - \frac{1}{2}x(t-t_0) = \mathrm{Re}\{(\tilde{x}(t) - \frac{1}{2}\tilde{x}(t-t_0)e^{-j2\pi f_c t_0})\cdot e^{j2\pi f_c t}\}$

$= \mathrm{Re}\{\tilde{y}(t)\cdot e^{j2\pi f_c t}\}$

$\tilde{y}(t) = \tilde{x}(t) - \frac{1}{2}\tilde{x}(t-t_0)\, e^{-j2\pi f_c t_0}$

$\tilde{Y}(f) = \tilde{X}(f) - \frac{1}{2}\tilde{X}(f)\cdot e^{-j2\pi f t_0}\cdot e^{-j2\pi f_c t_0}$

$\tilde{H}(f) = 1 - \frac{1}{2}e^{-j2\pi f t_0}\cdot e^{-j2\pi f_c t_0}$

$\Big\updownarrow \mathcal{F}^{-1}$

$\tilde{h}(t) = \delta(t) - \frac{1}{2}\delta(t-t_0)e^{-j2\pi f_c t_0}$

$h_b(t) = \frac{1}{2}\tilde{h}(t)$

$= \frac{1}{2}\left[\delta(t) - \frac{1}{2}\delta(t-t_0)e^{-j2\pi f_c t_0}\right]$

$\boxed{h_b(t) = \frac{1}{2}\left[\delta(t) - \frac{1}{2}\delta(t-t_0)e^{-j2\pi f_c t_0}\right]}$

but ~~it~~ h_b(t) is band limited to $[-W, W]$

5) (3 Marks) Let a real-valued wide sense stationary random process $X(t)$ be such that its autocorrelation function is periodic, i.e.,

$$\begin{aligned} R_X(\tau) &\triangleq \mathbb{E}[X(t)X(t-\tau)] \\ &= R_X(\tau+T) \end{aligned}$$

for some finite $T > 0$.

Prove that the random process $X(t)$ is also periodic with the same period T, i.e., $X(t) = X(t+T)$ with probability one (i.e., for some realization $x_w(t)$ of the random process $X(t)$ it might happen that $x_w(t) \neq x_w(t+T)$, but then the probability measure of all such realizations is 0).

Answer:

$$R_X(\tau) \triangleq E[X(t) \cdot X(t-\tau)]$$

$$R_X(\tau+T) \triangleq E[X(t) \cdot X(t-(\tau+T))]$$

$$P\{X(t) = X(t+T)\} = ?$$